Private Provision of a Complementary Public Good*

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Preliminary Version

Abstract

For several years, an increasing number of firms have begun to invest in open source software (OSS). While improvements in such a non-excludable public good are not appropriable, companies can benefit indirectly in a complementary proprietary segment. We study this incentive for investment in OSS. In particular we ask how (1) market entry and (2) public investments in the public good affects the firms' behaviour and profits. Surprisingly, we find that there exist cases where incumbents benefit from market entry. Moreover, we show the counter-intuitive result that public spending has not to result in decreasing voluntary private contribution.

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1 Introduction

For several years, an increasing number of firms like IBM and Hewlett-Packard or Suse and Red Hat have begun to invest in Open Source Software. Open Source Software, such as Linux, is typically under the General Public License (GPL) and therefore any improvement must be provided for free. Hence, an Open Source Software can be seen as a non-excludable public good meaning that firms are not able to sell the Open Source Software or their improvements. This issue raises the question why companies do contribute to such a public good.
Lerner and Tirole (2000) argue that firms expect to benefit from their expertise in some market segment of which the demand is boosted by the introduction of a complementary open source program. Although the companies cannot capture directly the value of an open source program's improvement they can profit indirectly through selling more complementary proprietary goods at a potentially higher price.

Notice, that this incentive to contribute to a non-excludable public good arises not only in the case of Open Source Software. In fact, if e.g. a firm's advertising increases not only the demand for the firm's own good but also the demand for its competitors' products (Friedman (1983) calls it cooperative advertising) or if a firm's lobby-activity has a positive effect on the whole industry, the analysis remains the same.

In this paper we study this incentive of investment in such a non-excludable public good. In particular we address the following two questions:

(1) What is the effect of a higher public good investment by the government on the firms' production and profits?

(2) How does market entry and therefore fiercer competition affect the incentive to contribute to the public good and how does it influence the incumbents' profits?

The first question is especially interesting because of the ongoing discussion if and how the government should support Open Source Software.[1] One can fear that an increase in the government's contribution to the public good decreases firms' voluntary spending like it is stated in the public good literature (e.g Bergstrom et al. (1985)). Interestingly, we will show that this does not have to be the case and that it is even possible that the firms' investments increase if the government rises its contribution to the public good. This means that it is not straight forward if the government's and the firms' investments are complements or substitutes to each other.

The second question gets shortly addressed by Lerner and Tirole (2000). They argue that the usual free-rider problem should appear because the firms are not able to capture all the benefits of their investments. Therefore, one might think that with an increasing number of firms the free-rider problem gets worse and so the firms' contribution to the public good decrease. Probably, as a consequence, this lower the firms' profits and the social surplus.
In this paper we show that the market entry of an additional firm has a positive externality (through the entrant's contribution to the public good) and a negative externality (through the entrant's production of the private good) on the incumbents.
We find that for certain cost and demand functions each firm reduces its output as a consequence of market entry and suffers a decrease in profits. In this case incumbents dislike market entry. Surprisingly, for certain cost and demand constellations, it is also possible that with market entry every firm expands its output and is able to increase its profit. However, a social planer unambiguously prefers market entry.

We contribute to answering the questions (1) and (2) by analyzing a model with Cournot-Competition where the firms can produce a private and a public good, but can sell only the private good. For the consumers, the private and public goods are complements. An increase in the available quantity of the public good increases their willingness to pay for the private good.

[1] see e.g. Hahn (2002), Evans and Reddy (2002) or Schmidt and Schnitzer (2002)

This article is related to the public good literature that is concerned with the private provision of a non-excludable public good. In standard models of public good provision,[2] households can buy or produce the private and the public good and they receive a certain utility directly from the existence of these goods.
In our model, firms do not benefit directly form the production of a public good. They produce the public good because of the complementarity to the private good. This leads to a new effect: The incentive of the firms to contribute to the public good depends on the market environment and therefore on the number of competitors. In the normal setup the marginal utility of the public good is determined through the utility function and is exogenously given. In our setup it is endogenous and depends on the ability to use the additional unit of the public good to earn money in the proprietary sector. This, however, depends on the degree of competition in this sector.

A second strand of literature our paper is related to is the literature of Multimarket Oligopoly. Bulow, Geanakoplos and Klemperer (1985) analyze the effects of a change in one market environment. In their model different markets are related through the cost function. In our model the markets are not related through the production technology but through the demand function. Bulow, Geanakoplos and Klemperer (1985) address this issue but do not formalize it. They mention that firms must take care off cross-effects in making marginal revenue calculations and consider the strategic effects of their actions in one market on competitors' actions in a second. In our model we formalize this issue and extend it to the interesting case of a non-excludable public good which is in contrast to the model of Bulow et al. (1985) where only private goods are considered.
Becker and Murphy (1993) analyze a model in which advertisement and an advertised good enters the utility function of the households. Advertisement has the property that it raises the willingness to pay for the advertised good

[2]see e.g. Bergstrom et al. (1986) for a general approach and e.g Bitzer and Schröder (2002) or Johnson (2002) for an application to the open source software development

and is from the economic viewpoint complementary to the advertised good. Nevertheless, in their setup a firm's advertisement is only complementary to its own private good. In our model it is also complementary to the competitors' private goods.
In some sense, the closed paper to ours is Friedman (1983). He uses a dynamic setup and treats advertising like capital in the sense that a firm can build up a "goodwill stock" through advertising comparable to a capital stock. Due to his modelling an increasing number of active firms leads to a steady-state in which conventional competitive effects dominate and prices converge to marginal costs. However, we look at a static game to concentrate explicitly on the externalities between the firms and the goods.

We will proceed as follows. The next section sets up the general model. In Section 3 we look at the properties of the market-equilibrium. After this we analyze in Section 4 the general production adjustment process and the reaction of the profits due to an exogenous change in the production of the private and the public good. In Section 5 we apply this analysis to determine the effects of a government intervention and the effects of market entry. The final section concludes.

2 The Model

Consumer

Consider individuals who consume two goods: A private good and a non-excludable public good. Let X describe the available quantity of the private good, e.g. hardware or service support, and Y the available quantity of the non-excludable public good, e.g. Open Source Software.[3] These two types

[3]We are always going to speak about quantity of the public good. In some cases one can interpret this quantity as a measure of quality.

of goods are complements for the consumers. The consumers have to pay a price p for every unit of the private good they want to consume. For the non-excludable public good there is by definition no price to pay. Therefore, every individual consumes the whole available quantity of the public good.

Firms

Firm i ($i \in \{1, 2, ..., N\}$) can produce the private good and the public good at costs of $K_i^x(X)$ and $K_i^y(Y)$. The production of the firm i's private good is denoted by x_i and its production of the public good by y_i. Firm i's revenue is $R_i = x_i * p(X, Y)$ and its profit function is $\pi_i = R_i(X, Y) - K_i^x(X) - K_i^y(Y)$.

Assumptions

In order to analyze the described problem we introduce the following assumptions:

(A1) $x_i \in X_i = [0; \overline{X_i}]$, $y_i \in Y_i = [0; \overline{Y_i}]$ and $s_i \in S_i = X_i \times Y_i$
(A2) $X = \sum_{i=1}^{N} x_i$ and $Y = \sum_{i=1}^{N} y_i$
(A3) $K_i^x(X)$ is convex in x_i and twice continuously differentiable with $K_i^x(X = 0) = 0$
(A4) $K_i^y(Y)$ is convex in y_i and twice continuously differentiable with $K_i^y(Y = 0) = 0$
(A5) The inverse demand function p is twice continuously differentiable and depends on X and Y with the properties: $p = p(X; Y)$, $\frac{\partial p}{\partial X} \leq 0$, $\frac{\partial p}{\partial Y} \geq 0$ and $p = p(X = 0; Y = 0) > 0$
(A6) R_i is twice continuously differentiable with $\frac{\partial R_i}{\partial x_i} \geq 0$, $\frac{\partial R_i}{\partial y_i} \geq 0$, $\frac{\partial^2 R_i}{\partial x_i \partial X} \leq 0$, $\frac{\partial^2 R_i}{\partial y_i \partial Y} \leq 0$ and $\frac{\partial^2 R_i}{\partial y_i \partial x_i} = \frac{\partial^2 R_i}{\partial x_i \partial y_i} = \frac{\partial^2 R_i}{\partial x_i \partial y_j} \geq 0, j \in \{1, ..., N\}/i$
(A7) $[\frac{\partial^2 R_i}{\partial x_i \partial y_i}]^2 < [\frac{\partial^2 R_i}{\partial x_i^2} - \frac{\partial^2 K_i^x(x_i)}{\partial x_i^2}] * [\frac{\partial^2 R_i}{\partial y_i^2} - \frac{\partial^2 K_i(^y y_i)}{\partial y_i^2}]$

Assumption (A1) restricts the strategy space S_i of a firm i to a nonempty convex subset of R^2. Therefore, the firms face a capacity constraint through

assuming infinite productions costs for $s_i \notin S_i$. Nevertheless, this assumption is without loss of generality because we always let $\overline{X_i}$ and $\overline{Y_i}$ be big enough so that a possible interior solution is achievable.
Assumption (A2) states the fact that only the current production can be consumed.
Assumptions (A3) and (A4) state that the firms' cost functions are convex in their own production decision. Furthermore, we assume that the production costs of a small quantity of X and Y are very low. In the next section we will describe further properties of the cost function in detail.
Assumption (A5) determines the properties of the inverse demand function. It is as usual non-increasing in X because the marginal utility of the private good is declining or constant. Since the public and private goods are complements, the consumers' willingness to pay for the private good is non-decreasing in Y. For an illustration think about a computer server (= the private good) and a Open-Source operating system (= the public good). The performance of the server depends crucially on the ability of the server operating system to use the power of the hardware. If the quality of the operating system is increasing without generating any costs for the consumers, their willingness to pay for the server increases due to the better performance or remains constant but never falls.
Furthermore we assume in Assumption (A5) that for certain range of X and Y there is a positive willingness to pay for the private good. Together with the Assumptions (A3) and (A5) this ensures that the firms have an incentive to produce.
Assumption (A6) determines that a firm's revenue increases in the private and the public good. Additional we assume declining marginal revenue of a good holding the other complementary good constant.
Furthermore, we assume that the marginal revenue of a firm's good increases with the firm's production level of the complementary good.
Assumption (A7) ensures that the firm's profit function is always concave and therefore has an unique maximum.

Time Structure

We assume that the firms are engaged in a one-period Cournot-Competition. Therefore, all firms decide simultaneously how much they produce of the private and the public good given the production decision of the others.

3 Properties of the Market Equilibrium

To determine the market equilibrium one has to think more precisely about the cost functions. For the private good like e.g computer hardware, one can easily make standard assumptions and use a convex cost function which is independent of the production decision of the others.

$$\frac{\partial K_i^x(x_i)}{\partial x_i} \geq 0, \frac{\partial^2 K_i^x(x_i)}{\partial x_i^2} \geq 0$$

The determination of the public good cost function properties is not so straight forward. On the hand one can argue that every firm has the same abilities and knowledge to improve an Open Source Software. This would mean that the costs for an additional improvement of the software are equal between the firms. For example, imagine that one firm already eliminated the easiest failures in a program. If all firms have the same abilities, this leads to the fact that the elimination of an additional failure would cause the same costs for every type of proceeding firm.
Translated into the cost function, this implies that the overall costs for the public good $K^y(Y) = \sum_{i=1}^{N} K_i^y(Y)$ depend only on the total production Y and is independent of the production distribution between these firms. Nevertheless, one has to decide how to split up the costs.
One possible assumption is that in the case of N firms every firm has to bear the fraction $\frac{1}{N}$ of the total costs. This would lead to a firm's cost function where its costs depend on the production of the other firms.

$$K_i^y(Y) = \frac{K^y(y_i + Y_{-i})}{N} \tag{1}$$

On the other hand one can bring forward the argument that the abilities and the knowledge of the firms are different. For example for Hewlett-Packard it may be very easy to improve a certain function of Linux dealing with printing which would in contrast be very demanding for IBM and the other way round. This implies that a firm's cost functions is independent of the other firms' production.

$$K_i^y(Y) = K_i^y(y_i) \tag{2}$$

However, we will show in the following Lemmas that what really matters is the property of a firm's marginal costs. In the case of **independent and increasing** marginal costs of the public good and with further technical assumption, we will show the existence of one unique market equilibrium. In contrast, if the public good's marginal costs are **dependent or constant** then there exist multiple of equilibria or corner solutions (Lemma 2 and 3).

Lemma 1

If the public good's marginal costs are independent and increasing then there exists a unique Nash Equilibrium where the firms produce $\nabla x^* = (x_1^*, x_2^*, ..., x_N^*)$ *of the private good and* $\nabla y^* = (y_1^*, y_2^*, ..., y_N^*)$ *of the public good if*

$$(3-N)*\frac{\partial p}{\partial x}+(2-N)(x_k*\frac{\partial^2 p}{\partial^2 x})+N*(x_k*\frac{\partial^2 p}{\partial x \partial y}+\frac{\partial p}{\partial y}) < \frac{\partial^2 K_k^x(x_k)}{\partial^2 x_k}, \forall k \in \{1, ..., N\}. \tag{3}$$

and

$$(N-1)|x_k*\frac{\partial^2 p}{\partial y \partial x}|+x_k*\frac{\partial^2 p}{\partial y \partial x}+\frac{\partial p}{\partial x}+(2-N)(x_k*\frac{\partial^2 p}{\partial^2 y}) < \frac{\partial^2 K_k^y(y)}{\partial^2 y_k}, \forall k \in \{1, ..., N\} \tag{4}$$

Proof:
see Appendix

To get the intuition behind this result, we write down the profit function of firm i

$$\pi_i = x_i * p\left(x_i, X_{-i}, y_i, Y_{-i}\right) - K_i^x(x_i) - K_i^y(y_i), \tag{5}$$

whereby

$$X_{-i} = \sum_{j=1, i \neq j}^{N} x_j \tag{6}$$

and

$$Y_{-i} = \sum_{j=1, i \neq j}^{N} y_j. \tag{7}$$

Profit maximization, given the production of all other companies, yields the following first order conditions for firm i

$$\frac{\partial \pi_i}{\partial x_i} = p + x_i \frac{\partial p}{\partial x_i} - \frac{\partial K_i^x(x_i)}{\partial x_i} = 0 \tag{8}$$

$$\frac{\partial \pi_i}{\partial y_i} = x_i \frac{\partial p}{\partial y_i} - \frac{\partial K_i^y(y_i)}{\partial y_i} = 0 \tag{9}$$

Equation (8) displays the standard condition for the optimal production decision of the private good where marginal costs equal marginal revenue. Equation (9) shows that the production of the public good has only an indirect effect on the profit. By raising the quantity of the public good, the consumers' willingness to pay increases, and therefore the firm achieves a higher price for their private goods.

Lemma 1 states that there is one unique Nash Equilibrium where the firms produce $\nabla x^* = (x_1^*, x_2^*, ..., x_N^*)$ and $\nabla y^* = (y_1^*, y_2^*, ..., y_N^*)$. To see this we proceed in three steps. First, we think about what is the optimal production decision with respect to the private good given the production of the public good. Second, we think about the optimal production decision with respect to the public good given the production of the private good. We will see that these decisions are always well-defined and unique. In the last step we show that starting form a point of no production we will arrive at an unique Nash Equilibrium.

First step: For the optimal production level of the private good x_i there are for firm i only two things important: The competitors' total production of the private good (like in the normal Cournot Game) and the total production of the public good Y. The last point results from the fact that for firm i's private good demand it does not matter if he or another firm produces one unit of the public good. Only the total production of the public good counts. For example, in order to sell its server hardware, for IBM its only important what is the quality of Linux and not who produced this quality. Furthermore, the technical assumptions ensure that the best reply function mapping with respect to the private good given the production of the public good is a contraction. This means that if you start at arbitrary vector $\nabla x^0 = (x_1^0, x_2^0, ..., x_N^0)$ and look at the best replies, you get a new vector $\nabla x^1 = (x_1^1, x_2^1, ..., x_N^1)$. If you look at the best replies to this vector and so on this sequence converges to a unique fixed point $\nabla x^* = (x_1^*, x_2^*, ..., x_N^*)$.

Second step: With the knowledge that every firm produces a certain quantity of x_i^* given Y we can now look at the production decision with respect to the public good y_i. To maximize a firm's profit, the marginal revenue with respect to the public good y_i must be equal to the marginal costs (see Equation 9). Due to the fact that the public good's marginal revenue is decreasing and its marginal costs are increasing, it is ensured that for every given production level of the private good $\nabla x^* = (x_1^*, x_2^*, ..., x_N^*)$ and the others' public good production Y_{-i}, there exists an unique optimal quantity y_i which firm i should produce. Furthermore, the second technical assumption in the Lemma ensures that the best reply functions with respect to the public good given the production of the private good are once again a contraction mapping. Therefore, as above, we have an unique Nash Equilibrium $\nabla y^* = (y_1^*, y_2^*, ..., y_N^*)$ given the individual production of the private good.

Third step: Now, we know that given a certain production of the private good ∇x (public good ∇y) there exists an unique Nash Equilirbium ∇y^* (∇x^*). The question is if we are able to find a unique ∇y^* which implies ∇x^* and the other way round. For this we start at a point where no firm produces anything and ask if this is a Nash Equilibrium. This is obvious not the case. Every firms has an incentive to produce at least some quantity of the private good x_i which follows immediately from Assumption (A3) and (A5). But this gives the firms an incentive to produce at least some amount of the public good y_i which once again gives an incentive to increase the production of the private good and so. The system converges to an unique interior Nash Equilibrium if the marginal costs increase fast enough which is stated in the two technical assumptions in Lemma 1. This is because the marginal revenues increase not fast enough to compensate for the higher marginal costs. In the end this leads to the unique Nash Equilibrium $\{(x_1^*, y_1^*), (x_2^*, y_2^*), ..., (x_N^*, y_N^*)\}$ where no deviation up or down pay-offs.

In the next two Lemmas we want to look at the case of constant or dependent marginal costs of the public good. We show that in these cases multiplicity of equilibria and corner solutions arise.

Lemma 2

If every firm has the same private good cost function and

- *if the public good's marginal costs are constant and equal for all firms or*
- *if the public good cost function is dependent on the others' public good production*

then there can be an infinite number of equilibria whereby in all equilibria x_i^* does not change and the firms' production of the public good y_i^* sums up to a certain constant level Y^*. The infinite number of equilibria arise if

$$(3-N)*\frac{\partial p}{\partial x}+(2-N)(x_k*\frac{\partial^2 p}{\partial^2 x})+N*(x_k*\frac{\partial^2 p}{\partial x \partial y}+\frac{\partial p}{\partial y}) < \frac{\partial^2 K_k^x(x_k)}{\partial^2 x_k}, \forall k \in \{1, ..., N\} \tag{10}$$

and

$$\exists i \in \{1, ...N\} s.t.\ x_i * \frac{\partial p(\sum_{j=1}^{N} \overline{Y_j}, \sum_{j=1}^{N} x_j^*)}{\partial y_i} < \frac{\partial K_i^y}{\partial y_i} \tag{11}$$

Proof:

see Appendix

What this case distinguishes from the first case is that given a certain total production of the of Y the marginal production costs of the public good are the same for every firm.

$$\frac{\partial^2 K_i^y}{\partial^2 y_i} = \text{const } \forall i \text{ given } Y \tag{12}$$

If the marginal costs are constant and for every firm the same this obviously true. In the case of dependent costs it is due to the fact that the total costs for producing the public good are divided equally among the firms which implies that every firm hast to bear $\frac{1}{N}$ of the the costs for an additional unit of the public good.

Once again, given a certain total production of the public good Y, there exists an unique Nash Equilibrium $\nabla x^* = (x_1^*, x_2^*, ..., x_N^*)$ like in the first step of independent and increasing marginal costs which is ensured by the first technical assumption in the Lemma.
What is new is that given every company produces a certain quantity of the private good x_i^* there does not exist an unique Nash Equilibrium $\nabla y^* = (y_1^*, y_2^*, ..., y_N^*)$. To see this, one has to realize that corner solutions (where every firm produces nothing of the public good or every firm produces until it reaches its capacity constraint) are ruled out by the technical assumptions (A5), (A7) and the second assumption of the Lemma 2. This implies that we have an interior solution whereby the firms' production of the public good sums up to a certain level of Y^*. Lemma 2 states that all production dividing rules with $\sum_{i=1}^{N} y_i = Y^*$ are Nash Equilibria.

Why was this not also the case in the Lemma 1? With independent and increasing marginal costs the individual production ∇y^* was clearly determined through the marginal increasing costs of producing the public good. These are now constant and therefore a coordination problem with an infinite amount of equilibria arises.
For an illustration suppose two firms which produce, in an Nash Equilibrium, the same quantities of the private good given Y. Assume that firm 1 thinks that firm 2 is going to produce nothing of the public good. Then firm 1 should produce until its marginal revenue is equal to its marginal costs given that the other firm produces nothing. In this case, it is indeed optimal for firm 2 to produce nothing because its marginal revenue and its marginal costs are the same as for the firm 1. If firm 1 is an optimum given the production of firm 2 then firm 2 is also in an optimum given the production of firm 1. And this fact leads to the infinite amount of possible equilibria and rises a coordination problem between the firms.

In the next Lemma we will show that it is possible that corner solutions arise. For simplicity we only address the case of two firms. Nevertheless, taking the first two Lemmas into account, one can easily find out the properties of the market equilibria with more than two firms.

Lemma 3
If

1.) the the two firms have the same private good cost functions and different constant marginal costs for producing the public good or

2.) if the two firms have different private good cost functions and constant marginal costs for producing the public good or

3.) if the two firms have different private good cost functions and dependent cost functions for producing the public good

and if the condition

$$(3-N)*\frac{\partial p}{\partial x}+(2-N)(x_k*\frac{\partial^2 p}{\partial^2 x})+N*(x_k*\frac{\partial^2 p}{\partial x \partial y}+\frac{\partial p}{\partial y}) < \frac{\partial^2 K_k^x(x_k)}{\partial^2 x_k}, \forall k \in \{1, ..., N\} \tag{13}$$

is fulfilled

then the firm i which would produce more of the public good given that the other firm produces nothing, will alone produce its optimal public good quantity y_i^*.

<u>**Proof:**</u>

see Appendix

To understand this result we first examine if the stated equilibrium is indeed a Nash Equilibrium. First realize that through our first technical assumption it is ensured (as in the cases before) that given Y there exists an unique Nash Equilibrium ∇x^*.
Now, assume that firm 1 would produce more of the public good given that the other firm produces nothing of the public good and denote this quantity by $y_1^*|_{y_2=0}$. It is obvious that firm 1 has no incentive to deviate because this quantity maximizes its profit function given that firm 2 produces nothing. Does firm 2 has an incentive to deviate by producing any quantity of the public good? It should deviate if its marginal revenue is bigger than the marginal costs of the public good because both firms produce the same quantity of the private good. In the Case 1 of the Lemma 3 both firms have the same marginal revenue with respect to the public good. If firm 2 would produce less of the public good given firm 1 produces nothing (we will call this $y_2^*|_{y_1=0}$) then it is obvious that the marginal revenue for firm 2 is smaller than the marginal costs given $y_1^*|_{y_2=0}$. Hence, firm 2 has no incentive to deviate and $\{(x_1^*, y_1^*|_{y_2=0} = y_1^*), (x_2^*, y_2^* = 0)\}$ is indeed a Nash Equilibrium.
In the Cases 2 and 3 of the Lemma 3 the two firms have different private good cost functions. This leads to an asymmetric quantity of the private

good x_i^* and therefore to different marginal revenue functions with respect to the public good $\frac{\partial R_i}{\partial Y}$. In the cases of constant and symmetric marginal costs of the public good or dependent marginal costs of the public good it is obvious that the equilibrium of Lemma 3 is indeed a Nash Equilibrium because the marginal revenue is different and the marginal costs (given a certain production of the public good) is the same. With different constant marginal costs and different marginal revenues one can see that if $y_1^*|_{y_2=0} > y_2^*|_{y_1=0}$ than for firm 2 the production of an additional unit of the public good has no value because its marginal revenue is smaller than its marginal costs given the production $y_1^*|_{y_2=0}$.

4 Change of the Incumbents' Profits and the Production Adjustment

In this section we want to make a quite general analysis and apply it in the next section. We therefore look on how one incumbent's profits and its production decision changes if the others' production of the public and private good changes for what reason whatsoever. We therefore assume that the firm are in a Nash-Equilibrium before the exogenous shock comes in. Furthermore we restrict our analysis to the immediate effects to a firm. So we do not consider how the adjustment of this firm influences the behavior of the other firms and how this once again influences the behavior of the firm and so on. In the last section we consider this case, whereby we have to use specific demand and cost functions to be able to make statements.

Change of Profits

Suppose that firm i faces a change in others' total production of the private good X_{-i} and the public good Y_{-i}. This is has pecuniary externalities on firm's i profits. An increase in the private good production X_{-i} has a negative pecuniary externality because it decreases the price for the private good in the market. An increase in the public good production Y_{-i} has an positive externality because it increases, due to the complementarity in the demand function, the price of the private good. These insights lead to the Proposition 1:

Proposition 1

If firm i faces a change in the total production X_{-i} of the private good and in the total production of Y_{-i} of the public good then the price of the private good in the market can change. If the change in X_{-i} and Y_{-i} is small and the new price is higher (lower) then the firm's profits increase (decrease). If the price does not change then the incumbent's profits are not affected.

Proof:

see Appendix

One may wonder if this analyze is really so simple. Indeed, a change in the production of the other firms has two effects on the profit. First it changes the price for the private good and therefore influences the profit directly. Second, the firm reacts to the new production level of the others and adjust its own production level which can also influence the firm's profit. Nevertheless, we know from the Envelope Theorem that as long as the changes are small this second effect is close to zero and could be neglected.

Adjustment of the Production

After considering the chances in the profit, we now want to look at the adjustment of the production decision.

This is not so easy as it looks like because one has to keep in mind cross effects. To see this, the easiest way is to distinguish between direct and indirect effects. Taking the total derivative of the firm's first order conditions one gets Equations 14 and 15.

Total derivative of $\frac{\partial \pi_i}{\partial x_i}$:

$$[2 * \frac{\partial p}{\partial x_i} + x_i \frac{\partial^2 p}{\partial x_i^2} - \frac{\partial^2 K(x_i)}{\partial x_i^2}]dx_i + \overbrace{[\frac{\partial p}{\partial y_i} + x_i \frac{\partial^2 p}{\partial x_i \partial y_i}]dy_i}^{\text{indirect effect}} \tag{14}$$

$$\overbrace{+[\frac{\partial p}{\partial X_{-i}} + x_i \frac{\partial^2 p}{\partial x_i \partial X_{-i}}]dX_{-i} + [\frac{\partial p}{\partial Y_{-i}} + x_i \frac{\partial^2 p}{\partial x_i \partial Y_{-i}}]dY_{-i}}^{\text{direct effects}} = 0$$

Total derivative of $\frac{\partial \pi_i}{\partial y_i}$:

$$\begin{gathered}\overbrace{[\frac{\partial p}{\partial y_i} + x_i \frac{\partial^2 p}{\partial x_i \partial y_i}]dx_i}^{\text{indirect effect}} + [x_i \frac{\partial^2 p}{\partial y_i^2} - \frac{\partial^2 K_i(y_i)}{\partial y_i^2}]dy_i \\ \overbrace{+[x_i \frac{\partial^2 p}{\partial y_i \partial X_{-i}}]dX_{-i} + [x_i \frac{\partial^2 p}{\partial y_i \partial Y_{-i}}]dY_{-i}}^{\text{direct effects}} = 0\end{gathered} \tag{15}$$

Equation (14) and (15) show that we can decompose the overall effect into direct and indirect effects in order to understand the complex adjustment process.
1. The direct effect influences x_i (y_i) through a change in X_{-i} or Y_{-i} without depending on a change in the corresponding complement y_i (x_i).
2. The indirect effect influences the optimal level of x_i (y_i) through a change in the corresponding complement y_i (x_i).

Suppose for an illustration that IBM faces an exogenous increase in the quality of the Linux. On the one hand this probably increases the consumers' valuation of IBM's server and therefore gives IBM an incentive to increase its production of the hardware (= a direct effect of Y_{-i} on x_i). On the other hand the additional supply of the public good might decrease the firm's production of the public good because keeping the old production level would lead to a higher overall amount of the public good (= a direct effect of Y_{-i} on y_i). Furthermore, there are the indirect effects. If IBM produces more server this changes probably the incentive to invest in the public good Linux (=indirect effect of x_i on y_i). Additional if IBM changes its investment in Linux, the quality of the operation system changes and this will have an effect on the incentives to produce the private good server (= indirect effect of y_i on x_i).

To isolate the different effects one can apply the implicit function theorem and see how x_i^* and y_i^* react on change in X_{-i} or Y_{-i}. These reactions are summarized in Lemma 4.

Lemma 4

If due to an exogenous shock the private good production X_{-i} changes then the effect on firm's i production of the private and the public good is determined through:

$$\frac{\partial x_i^*}{\partial X_{-i}} = \frac{-\partial f^1/\partial X_{-i} * \partial f^2/\partial y_i^* + \partial f^2/\partial X_{-i} * \partial f^1/\partial y_i^*}{\partial f^1/\partial x_i^* * \partial f^2/\partial y_i^* - \partial f^1/\partial y_i^* * \partial f^2/\partial x_i^*} \tag{16}$$

$$\frac{\partial y_i^*}{\partial X_{-i}} = \frac{-\partial f^1/\partial x_i^* * \partial f^2/\partial X_{-i} + \partial f^2/\partial x_i^* * \partial f^1/\partial X_{-i}}{\partial f^1/\partial x_i^* * \partial f^2/\partial y_i^* - \partial f^1/\partial y_i^* * \partial f^2/\partial x_i^*} \tag{17}$$

with $f^1 = \frac{\partial \pi_i}{\partial x_i}$ *and* $f^2 = \frac{\partial \pi_i}{\partial y_i}$.

If due to an exogenous shock the public good production Y_{-i} changes then the effect on firm's i production of the private and the public good is determined through:

$$\frac{\partial x_i^*}{\partial Y_{-i}} = \frac{-\partial f^1/\partial Y_{-i} * \partial f^2/\partial y_i^* + \partial f^2/\partial Y_{-i} * \partial f^1/\partial y_i^*}{\partial f^1/\partial x_i^* * \partial f^2/\partial y_i^* - \partial f^1/\partial y_i^* * \partial f^2/\partial x_i^*} \tag{18}$$

$$\frac{\partial y_i^*}{\partial Y_{-i}} = \frac{-\partial f^1/\partial x_i^* * \partial f^2/\partial Y_{-i} + \partial f^2/\partial x_i^* * \partial f^1/\partial Y_{-i}}{\partial f^1/\partial x_i^* * \partial f^2/\partial y_i^* - \partial f^1/\partial y_i^* * \partial f^2/\partial x_i^*} \tag{19}$$

with $f^1 = \frac{\partial \pi_i}{\partial x_i}$ *and* $f^2 = \frac{\partial \pi_i}{\partial y_i}$.

Proof:

see Appendix

In the next section we want to apply this Lemma 4 to analyze the effect of government intervention.

5 The Analysis of Government Intervention and the Effects of Market Entry

In this section we first want to analyze what happens if the government engages in the production of the public good. A Government intervention is not a hypothetical question. For example, the U.S. government has supported the development of open-source software like Linux[4]. In a second step we want to consider the case of an market entry and determine the effects on the incumbents' production and profits.

Government Intervention

If the Government increases its contribution to the public good, the effect on an incumbent firm's profit is straightforward. The additional supply of the public good increases the price of the private good and therefore, as stated in Proposition 1, the profits increase.

What is not so obvious is the adjustment of a firm's production. The standard public good literature states that if the government increases its investment in a public good, like the social security system, then the agents usually decrease their donations to the this public good[5]. We will show that in our context this have not to be true and that even the opposite may happen.

At first we want to look at the case of a total crowding out, meaning that an incumbent firm would reduce its production of the public good by 1:1 if the government increases its investment of the public good. Furthermore, the incumbent does not adjust its production of the private good.

[4] Evans and Reddy (2002) p.2

[5] see e.g. Bergstrom, Blume and Varian (1985)

Proposition 2

If

- *the marginal revenue of the private good is increasing with respect to the public good* ($\frac{\partial^2 R}{\partial x \partial y} > 0$)
- *the marginal revenue of the public good is decreasing* ($\frac{\partial^2 R}{\partial y^2} < 0$)
- *the marginal production costs of the public good are constant* ($\frac{\partial^2 K}{\partial y^2} = 0$)

then an increase in the government investment in the public good leads to a 1:1 reduction of the firm's investment in the public good without changing the production level of the private good.

Proof:

see Appendix

Due to the fact that the marginal costs of the public good are constant and the marginal revenue of the public good is decreasing, there is an intersection where the marginal revenue is equal to the marginal cost. This point determines my optimal production level of the public good. If now the state increases the production of the public good, holding everything else constant, then the marginal costs are higher than the marginal revenue, because the marginal revenue is decreasing. Therefore the firm gets an incentive to decrease its production of the public good. It should decrease its production until the marginal costs are equal marginal revenue, once again. And this is achieved by a 1:1 reduction. In the end, the total available amount of the public good does not change and this leads to no adjustment of the private good production. Form a welfare point-of-view the consumers do not benefit form the government intervention. Only the incumbent firm gets better off. An intervention under this circumstances is like transferring money from the government to the firm.

The analysis completly changes if we assume that the marginal revenue of the public good is constant and the marginal costs of the public good are increasing.

Proposition 3

If

- *the marginal revenue of the private good is increasing with respect to the public good (*$\frac{\partial^2 R}{\partial x \partial y} > 0$*)*
- *the marginal revenue of the public good is constant (*$\frac{\partial^2 R}{\partial y^2} = 0$*)*
- *the marginal production costs of the public good are increasing (*$\frac{\partial^2 K}{\partial y^2} > 0$*)*

then an increase in the government investment in the public good leads to an increase in the production of the private and the public good.

Proof:

see Appendix

In this case the public good's marginal costs are increasing and the marginal revenue of the public good is constant. This leads to an intersection where the marginal costs are equal to the marginal benefit, determining the optimal production level of the public good given the production of the private good. If now the state increases its contribution to the public good, then this does not change the marginal revenue with respect to the public good. Therefore an incumbent has no direct incentive to adjust its production of the public good. But, because the whole available amount of the public increases the incumbent has an incentive to increase its production of the private good (=direct effect), because the private good's price raises. This increase in the private good's production has an feedback-effect (= indirect effect) on the incentives to produce the public good, because the marginal revenue of public good is increasing in the production of the private good. In the end this leads to a higher production of the incumbent and increases the available amount of the private and the public good.

One can look at further cases. But this analysis should be relegated to the appendix, because theses tow cases highlight the basic forces in such a market.

Effects of Market Entry

We now want to consider the effects of market entry. Since yet, we have assumed that an incumbent faces an exogenous shock in the production of the public good. With market entry this exogenous shock is determined endogenously. The entrant will choose an optimal production level of private and the public good to maximize its profits. Thereby, the new firm takes into account how the incumbents will react and change their production. To be able to analyze such a case, we restrict us to the case of symmetric firms with quadratic cost functions and facing a linear-demand function. Nevertheless, we are able to point out the crucial issues.

Therefore, we assume an inverse demand function which is linear in x and y.

$$p = A - X + Y \tag{20}$$

Consider N symmetric firms and assume for all firms quadratic cost functions.

$$K(x) = dx^2 \tag{21}$$

$$K(y) = fy^2 \tag{22}$$

The parameters d and f represent the weight of the cost functions with which they influence the firm's profit.

We know, due to Proposition 1, that in an equilibrium all competitors produce the same amount of the private good x and the same amount of the public good y. Thus profits for firm i, whereby x_j and y_j represents the amount produced by firm $j \in \{1, ..., N\}$ with $j \neq i$, becomes

$$\pi_i = x_i * (A - x_i - (N-1)x_j + y_i + (N-1)y_j) - dx_i^2 - fy_i^2 \tag{23}$$

The first-order conditions are

$$\frac{\partial \pi_i}{\partial x_i} = A - 2x_i - (N-1)x_j + y_i + (N-1)y_j - 2dx_i = 0 \tag{24}$$

$$\frac{\partial \pi_i}{\partial y_i} = x_i - 2fy_i = 0 \tag{25}$$

Solving (25) for y_i^* yields to

$$y_i^* = \frac{1}{2f}x_i \tag{26}$$

Equation (26) shows that y_i^* depends only on the amount produced of the firm's private good and on the weight of the public good's cost function f. Therefore, the optimal level of y_i^* is independent of the production decision of the other firms and changes only if x_i^* changes. We summarize this observation in the following Lemma.

Lemma 5

Each firm produces the private good x_i^* and the public good y_i^* in the same ratio which is determined by the weight of the public good cost function on the firm's profit f.

$$y_i^* = \frac{1}{2f}x_i$$

Proof:

see Appendix

The intuition for this is as follows: For a firm the production of a public good has the only effect of increasing the price of the private good. In our case the effect of y_i on p is always constant and equal to 1. The marginal revenue of y_i is therefore x_i. The marginal costs are $2fy_i$. In the optimum the marginal revenue must equal the marginal costs. Therefore, we can see that the relationship between x_i and y_i is linear and the constant slope depends only on the weight of the public good's production cost f.

Next we want to determine the optimal amount of the private good for a firm. Solving (24) for x_i^* and using (26) and symmetry yields to

$$x_i^* = \frac{A}{1 + N - \frac{1}{2f}N + 2d} \tag{27}$$

To get economically reasonable values we restrict the possible values of the numbers of firms N in such a way that x_i^* cannot be negative which can only be the case if the weight of the public good's cost function is low ($f < 0.5$). In this case we restrict N to be smaller than $\overline{N}$ with

$$\overline{N} = \frac{2d + 1}{\frac{1}{2f} - 1} \tag{28}$$

For an optimum it is necessary and sufficient that the second-order conditions are fulfilled.

$$\frac{\partial^2 \pi}{\partial x^2} = -2 - 2d < 0 \tag{29}$$

$$\frac{\partial^2 \pi}{\partial y^2} = -2f < 0 \tag{30}$$

$$[\frac{\partial^2 \pi}{\partial x \partial y}]^2 = 1 < (-2 - 2d)(-2f) \tag{31}$$

After deriving the firms' optimal production decisions we want to look how these change due to market entry.
Therefore, suppose that an additional firm enters the market. With symmetry the new firm can use the same production technology as the incumbents. The new firm is going to produce the private and the public good. This has two effects: On the one hand competition increases due to the additional production of the private good which may lower the incentives to produce x_i. On the other hand the entrant's production of the public good raises the consumers' valuation of the private good and therefore gives an incentive to increase the production of the private good. Hence, it is not obvious in which direction an incumbent is going to adjust its production of the private good.

Proposition 4

If the number of competing firms N increases then

- *each incumbent reduces his production of the private good x_i^* if the weight of the public good's cost function on the profit ensures that the entrant produces more of the private good than of the public good ($f > 0.5$)*
- *each incumbent does not change his production of the private good x_i^* if the weight of the public good's cost function ensures that the entrant produces as much of the public good as of the private good ($f = 0.5$)*
- *each incumbent increases his production of the private good x_i^* if the weight of the public good's cost function on the profit ensures that the entrant produces more of the public good than of the private good ($f < 0.5$)*

Proof:

To proof Proposition 4 we take the first derivative of x_i^* with respect to N.

$$\frac{\partial x}{\partial N} = -\frac{A}{[1 + N - \frac{1}{2f}N + 2d]^2} * (1 - \frac{1}{2f}) \tag{32}$$

We can see that the sign is negative if $f > 0.5$, is zero if $f = 0.5$ and is positive if $f < 0.5$.

Q.E.D.

Proposition 4 states the somehow surprising result that the effect on x_i^* depends only on the weight of the public good's cost function. This issue becomes clear if we consider the effects of a new firm on the incumbents. In the equilibrium the new firm produces the same amount of the public and of the private good as every incumbent after the adjustment process. Due to the chosen demand function, x and y influence the price p always with the same weight. Therefore, if in the equilibrium the additional competitor produces the same amount of the private good as of the public good, the firm does not influence the behavior of the incumbents. This is the case because the entrant does not alter the price and therefore in some way the incumbents do not even notice the market entry.

If the weight of the public good's cost function is small ($f < 0.5$) then every firm produces more of the public good than of the private good. This increases the price. Without altering their production the incumbents get a higher price for their last produced unit of the private good. Since the costs for the production did not change the incumbents should expand their production so that in the end the usual condition for an optimum "marginal revenue equals marginal cost" is fulfilled. If the weight of the public good's cost function is high ($f > 0.5$) then the opposite is true.

Using Proposition 5 and 4 we can can determine the shift of the public good production of a firm y_i.

Proposition 5

If the number of competing firms N increases then

- *each incumbent reduces his production of the public good y_i^* if due to the market entry it lowers its production of the private good*
- *each incumbent does not change his production of the public good y_i^* if due to the market entry it does not change its production of the private good*
- *each incumbent increases his production of the public good y_i^* if due to the market entry it raises its production of the private good*

Proof:

see Appendix

Proposition 4 states under which circumstances every firm increases or decreases its production of the public good or holds it constant. Due to Equation (26) we know that for every firm the optimal amount of the public good is determined through its production of the private good because the production of the public good has just the effect of raising the achievable price for the private good. Therefore, it is straightforward that the adjustment of the amount of the public good gets in the same direction as the adjustment of the production of the private good. This is the case because due to higher amount produced of the private good the marginal revenue of the public good increases. Especially, this is true since the effect of the public good on the price of the private good is always constant and therefore independent of a change in the total produced amount of the private and of the public good.

Next we want to look at the variation of the total amount of the private and public good. By Proposition 4 and by Proposition 5 we know that under some circumstances every firm decreases its production of the private and the public good. Therefore it is questionable if the production of the entrant can compensate this decline of the old producers' production.

Proposition 6

Market entry unambiguously increases the total amount of the private good X and the public good Y.

Proof:

see Appendix

Therefore, we see that the production of the additional firm is always big enough to compensate for the loss of production by the incumbents. This implies an interesting consequence:
The outcome of a normal Cournot-Game with the number of firms progressing towards infinity is equivalent to the outcome of a game with perfect competition where the firms behave as price takers.
In our setup the production of the public good should break down if the firms behave as price takers because the motivation for the production of the public good is the change of the private good's price. Hence, in our model the market equilibrium of a Cournot-Game with a infinite number of firms is no longer equivalent to the market equilibrium where the firms behave as price taker because the equilibria are different.

With the knowledge about the change of the total amount of the private and the public good we can determine the variation of the price.

Proposition 7

If the number of competing firms N increases then

- *the price p decreases if every firm produces more of the private good than of the public good ($f > 0.5$)*
- *the price p does not change if every firm produces as much of the private good as of the public good ($f = 0.5$)*
- *the price p increases if every firm produces more of the public good than of the private good ($f < 0.5$)*

Proof:

see Appendix

The result gets intuitively clear if one takes into account the total change of the amount produced. We know by Proposition 5 that the ratio between the public and private good is always constant. This means that the absolute difference between the total amount of the private and public good is increasing with the number of active firms because the total production of both goods is rising when more firms produce. Therefore, if for example every firm produces more of the private good than of the public good market entry leads to a price reduction because every unit of the private and every unit of the public good influences the price with the same weight and the difference between the total produced amount of both goods increases. Only in the case where every firm produces as much of the private good as of the public good the price does not change due to market entry since the negative effect on the price through the public good is fully compensated through the positive effect on the price through the private good.

After analyzing the change of the amounts produced and the price we now want to look at the profits of the firms and the social surplus.

In a normal Cournot-Competition, i.e. without a public good and only with a private good, the incumbents dislike market entry since the entrant has a negative pecuniary externality through the additional supply of the private good on the already producing firms. Now in our setup the entrant may have also a positive pecuniary externality since the entrant also contributes to the public good.

Proposition 8

If the number of competing firms N increases then

- *the profit of the incumbents π_i decreases if every firm produces more of the private good than of the public good ($f > 0.5$)*
- *the profit of the incumbents π_i does not change if every firm produces as much of the private good as of the public good ($f = 0.5$)*
- *the profit of the incumbents π_i increases if every firm produces more of the public good than of the private good ($f < 0.5$)*

Proof:

see Appendix

By Proposition 4 and by Proposition 5 we know that with market entry and a high weight of the public good's cost function ($f > 0.5$) every firm produces less of both goods and that the price in the market decreases. Hence, it is straightforward that under these circumstances, the profit of the incumbents decrease, which is the usual effect of stronger competition. If $f < 0.5$, we get the surprising result that the incumbents prefer more competition. This is due to the fact that a symmetric competitor does not only produce the private good which has a pecuniary negative external effect on the incumbents, but also produces some amount of the public good and therefore has a pecuniary positive external effect on the incumbents. If the weight of the production cost of the public good is small, then the positive external effect of a market entry dominates over the negative effect.

One might think that the social surplus reacts ambiguously to a market entry because sometimes the firms get higher profits and the price for the private good rises. But this is not the case.

Proposition 9

If the number of competing firms N increases then the social surplus always gets higher.

Proof:

see Appendix

If we just look at the consumer surplus we see that it is increasing with the number of active firms. This can be explained by two effects. Firstly, due to market entry the production of the public good increases. This has the effect that every consumer values the private good higher which has a positive effect on the consumer surplus. Secondly, the market entry leads to a tougher competition in the proprietary sector which increases the total production and has once again a positive effect on the consumer surplus. So even if the market entry leads to higher prices we get a higher consumer surplus. In our setup the relevant issue is the total produced amount of both goods. Since the firms cannot make any price discrimination a higher produced amount of the goods lead to a higher consumer surplus.

Proposition 8 shows that with a low weight of the public good's production cost market entry increases the profits of the firms. In this case it is obvious that the total surplus also increases. Nevertheless, if the firms' profit decrease through the market entry the gain in the consumer surplus compensates this effect and results in a higher total surplus. Therefore, a social planer always prefers market entry.

6 Conclusion

In this paper we have studied the incentive to contribute to a non-excludable public good if it is complementary to a private good.
We have shown that an intervention of the government in the public good production can lead to different results. It could be that the firms decrease their production and that a crowding-out occurs. On the other hand it is possible that the firms increase their production level of the private and the public good. So one has to be very carefully in forecasting the effect and has to look very closely at the revenue and cost functions of the active firms.
Furthermore we have looked at the case of market entry. We showed that an entrant has positive and negative pecuniary externalities on the incumbents. This could lead to the surprising effect, that under some circumstances an incumbent like market entry.

Appendix

Proof of Lemma 1:

We proceed in two steps. First we prove existence and afterwards uniqueness.

1. Existence:

From the Assumptions A2 follows that the strategy spaces $S_i = X_i \times Y_i$ with $X_i \in [0; \overline{X_i}]$ and $Y_i \in [0; \overline{Y_i}]$ are nonempty compact convex subsets of R^2. Furthermore by Assumptions A3, A4, A6 and A7 the profit functions π_i are continuous in s and quasi-concave in s_i. Therefore, if follows immediately that there exists an pure-strategy Nash equilibrium (Debreu, 1952).

2. Uniqueness

To show uniqueness we apply the contraction mapping approach. Due to Bertsekas (1999) it is sufficient to show that the Hessian of the profit functions fulfills the "diagonal dominance" condition.

$$H = \begin{vmatrix} \frac{\partial^2\pi_1}{\partial x_1\partial x_1} & \frac{\partial^2\pi_1}{\partial x_1\partial y_1} & \frac{\partial^2\pi_1}{\partial x_1\partial x_2} & \frac{\partial^2\pi_1}{\partial x_1\partial y_2} & \cdots & \frac{\partial^2\pi_1}{\partial x_1\partial x_n} & \frac{\partial^2\pi_1}{\partial x_1\partial y_n} \\ \frac{\partial^2\pi_1}{\partial y_1\partial x_1} & \frac{\partial^2\pi_1}{\partial y_1\partial y_1} & \frac{\partial^2\pi_1}{\partial y_1\partial x_2} & \frac{\partial^2\pi_1}{\partial y_1\partial y_2} & \cdots & \frac{\partial^2\pi_1}{\partial y_1\partial x_n} & \frac{\partial^2\pi_1}{\partial y_1\partial y_n} \\ \frac{\partial^2\pi_2}{\partial x_2\partial x_1} & \frac{\partial^2\pi_2}{\partial x_2\partial y_1} & \frac{\partial^2\pi_2}{\partial x_2\partial x_2} & \frac{\partial^2\pi_2}{\partial x_2\partial y_2} & \cdots & \frac{\partial^2\pi_2}{\partial x_2\partial x_n} & \frac{\partial^2\pi_2}{\partial x_2\partial y_n} \\ \frac{\partial^2\pi_2}{\partial y_2\partial x_1} & \frac{\partial^2\pi_2}{\partial y_2\partial y_1} & \frac{\partial^2\pi_2}{\partial y_2\partial x_2} & \frac{\partial^2\pi_2}{\partial y_2\partial y_2} & \cdots & \frac{\partial^2\pi_2}{\partial y_2\partial x_n} & \frac{\partial^2\pi_2}{\partial y_2\partial y_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2\pi_n}{\partial x_n\partial x_1} & \frac{\partial^2\pi_n}{\partial x_n\partial y_1} & \frac{\partial^2\pi_n}{\partial x_n\partial x_2} & \frac{\partial^2\pi_n}{\partial x_n\partial y_2} & \cdots & \frac{\partial^2\pi_n}{\partial x_n\partial x_n} & \frac{\partial^2\pi_n}{\partial x_n\partial y_n} \\ \frac{\partial^2\pi_n}{\partial y_n\partial x_1} & \frac{\partial^2\pi_n}{\partial y_n\partial y_1} & \frac{\partial^2\pi_n}{\partial y_n\partial x_2} & \frac{\partial^2\pi_n}{\partial y_n\partial y_2} & \cdots & \frac{\partial^2\pi_n}{\partial y_n\partial x_n} & \frac{\partial^2\pi_n}{\partial y_n\partial y_n} \end{vmatrix} \tag{33}$$

Therefore the diagonal of the Hessian dominates the off-diagonal entries if

$$\sum_{i=1, i\neq k}^{n} |\frac{\partial^2\pi_k}{\partial x_k\partial x_i}| + \sum_{i=1}^{n} |\frac{\partial^2\pi_k}{\partial x_k\partial y_i}| < |\frac{\partial^2\pi_k}{\partial x_k\partial x_k}|, \forall k \tag{34}$$

and

$$\sum_{i=1}^{n} |\frac{\partial^2 \pi_k}{\partial y_k \partial x_i}| + \sum_{i=1, i\neq k}^{n} |\frac{\partial^2 \pi_k}{\partial y_k \partial y_i}| < |\frac{\partial^2 \pi_k}{\partial y_k \partial y_k}|, \forall k \tag{35}$$

is fulfilled.

Calculating the derivatives and plugging into Equation 34 leads to

$$\sum_{i=1, i\neq k}^{N} |x_k * \frac{\partial^2 p}{\partial x_k \partial x_i} + \frac{\partial p}{\partial x_i}| + \sum_{i=1}^{N} |x_k * \frac{\partial^2 p}{\partial x_k \partial y_i} + \frac{\partial p}{\partial y_i}| < |\frac{\partial^2 p}{\partial x_k \partial x_k} * x_k + 2 * \frac{\partial p}{\partial x_k} - \frac{\partial^2 K_k(x)}{\partial x_k \partial x_k}|, \forall k \tag{36}$$

$$(N-1) * |(x_k * \frac{\partial^2 p}{\partial x_k \partial x_i} + \frac{\partial p}{\partial x_i}| + N * |x_k * \frac{\partial^2 p}{\partial x_k \partial y_i} + \frac{\partial p}{\partial y_i}| < |\frac{\partial^2 p}{\partial x_k \partial x_k} * x_k + 2 * \frac{\partial p}{\partial x_k} - \frac{\partial^2 K_k(x)}{\partial x_k \partial x_k}|, \forall k \tag{37}$$

$$(1-N)(x_k * \frac{\partial^2 p}{\partial^2 x} + \frac{\partial p}{\partial x}) + N * (x_k * \frac{\partial^2 p}{\partial x \partial y} + \frac{\partial p}{\partial y}) < -(\frac{\partial^2 p}{\partial^2 x} * x_k + 2 * \frac{\partial p}{\partial x}) + \frac{\partial^2 K_k(x_k)}{\partial^2 x_k}, \forall k \tag{38}$$

$$(2-N)(x_k * \frac{\partial^2 p}{\partial^2 x} + \frac{\partial p}{\partial x}) + N * (x_k * \frac{\partial^2 p}{\partial x \partial y} + \frac{\partial p}{\partial y}) < -(\frac{\partial p}{\partial x}) + \frac{\partial^2 K_k(x_k)}{\partial^2 x_k}, \forall k \tag{39}$$

$$(3-N) * \frac{\partial p}{\partial x} + (2-N)(x_k * \frac{\partial^2 p}{\partial^2 x}) + N * (x_k * \frac{\partial^2 p}{\partial x \partial y} + \frac{\partial p}{\partial y}) < \frac{\partial^2 K_k(x_k)}{\partial^2 x_k}, \forall k \tag{40}$$

Calculating the derivatives and plugging into Equation 35 leads to

$$\sum_{i=1, i\neq k}^{n} |x_k * \frac{\partial^2 p}{\partial y_k \partial x_i}| + |x_k * \frac{\partial^2 p}{\partial y_k \partial x_k} + \frac{\partial p}{\partial x_k}| + \sum_{i=1, i\neq k}^{n} |x_k * \frac{\partial^2 p}{\partial y_k \partial y_i}| < |x_k * \frac{\partial^2 p}{\partial y_k \partial y_k} - \frac{\partial^2 K_k(y)}{\partial y_k \partial y_k}|, \forall k \tag{41}$$

$$(N-1)|x_k*\frac{\partial^2 p}{\partial y_k \partial x_i}|+|(x_k*\frac{\partial^2 p}{\partial y_k \partial x_k}+\frac{\partial p}{\partial x_k}|+(N-1)|x_k*\frac{\partial^2 p}{\partial y_k \partial y_i}| < |x_k*\frac{\partial^2 p}{\partial y_k \partial y_k}-\frac{\partial^2 K_k(y)}{\partial y_k \partial y_k}|, \forall k \tag{42}$$

$$(N-1)|x_k*\frac{\partial^2 p}{\partial y_k \partial x_i}|+x_k*\frac{\partial^2 p}{\partial y_k \partial x_k}+\frac{\partial p}{\partial x_k}+(1-N)(x_k*\frac{\partial^2 p}{\partial y_k \partial y_i}) < -(x_k*\frac{\partial^2 p}{\partial y_k \partial y_k})+\frac{\partial^2 K_k(y)}{\partial y_k \partial y_k}, \forall k \tag{43}$$

$$(N-1)|x_k*\frac{\partial^2 p}{\partial y_k \partial x_i}|+x_k*\frac{\partial^2 p}{\partial y_k \partial x_k}+\frac{\partial p}{\partial x_k}+(2-N)(x_k*\frac{\partial^2 p}{\partial y_k \partial y_i}) < \frac{\partial^2 K_k(y)}{\partial y_k \partial y_k}, \forall k \tag{44}$$

$$(N-1)|x_k*\frac{\partial^2 p}{\partial y \partial x}|+x_k*\frac{\partial^2 p}{\partial y \partial x}+\frac{\partial p}{\partial x}+(2-N)(x_k*\frac{\partial^2 p}{\partial y \partial y}) < \frac{\partial^2 K_k(y)}{\partial^2 y_k}, \forall k \tag{45}$$

Q.E.D.

<u>Proof of Lemma 2:</u>

We proceed in two steps. First we prove existence and show afterwards that multiplicity is possible.

1. Existence

see Lema 1

2. Multiplicity

We proceed in 3 steps:

Step 1:
Due to the first technical assumption in the Lemma 2 we have, given $(y_1, y_2, ...y_N)$, a contraction mapping of the best reply functions with respect to the private

good. This leads to a unique Nash Equilibrium $\nabla x^* = (x_i^*, x_2^*, ..., x_N^*)$.

Step 2:
By Assumptions (A4),(A5) and the second technical assumption in the Lemma 2 the Nash Equilibrium cannot be

$$s^* = ((x_1^*, y_1^* = 0), (x_2^*, y_2^* = 0), ..., (x_j^*, y_j^* = 0), (x_k^*, y_k^* = 0), ..., (x_N^*, y_N^* = 0))$$

or

$$s^* = ((x_1^*, \overline{Y_1}), (x_2^*, \overline{Y_2}), ..., (x_j^*, \overline{Y_j}), (x_k^*, \overline{Y_k}), ..., (x_N^*, \overline{Y_N})).$$

Step 3:
By the proof of existence we know that there exists at least one Nash Equilibrium. Therefore, we can assume that

$$s^* = ((x_1^*, y_1^*), (x_2^*, y_2^*), ..., (x_j^*, y_j^*), (x_k^*, y_k^*), ..., (x_N^*, y_N^*))$$

is a Nash Equilibrium.

By Step 2 it is possible to find a $y_j \in]0; \overline{Y_j}[$ and a $y_k \in [0; \overline{Y_k}]$. Furthermore, it is possible to find a $\mu \in \Re$ s.t. $y'_j = y_j + \mu \in [0; \overline{Y_j}] \wedge y'_k = y_k - \mu \in [0; \overline{Y_k}]$.

s^* implies that the FOCs of every firm $i \in \{1, ..., N\}$ must be fulfilled at the values of s^*:

$$\frac{\partial \pi_i}{\partial x_i} = \frac{\partial R_i}{\partial x_i} - \frac{\partial K_i^x}{\partial x_i} = 0$$

$$\frac{\partial \pi_i}{\partial y_i} = \frac{\partial R_i}{\partial y_i} - \frac{\partial K_j^y}{\partial y_i} = 0$$

<u>Lemma:</u> If this is the case then these first order conditions are also fulfilled with the values $s' = ((x_1^*, y_1^*), (x_2^*, y_2^*), ..., (x_j^*, y'_j), (x_k^*, y'_k), ..., (x_N^*, y_N^*))$.

<u>Proof:</u> $Y = \sum_{i=1}^{k} y_i$ does not change. By

$$\frac{\partial^2 K_i^y}{\partial^2 y_i} = \text{const } \forall i \text{ given } Y$$

the marginal costs of all firms remain the same. Furthermore, the firms' marginal revenue do not change. This leads to the conclusion that all first order conditions are fulfilled at s'.

If all first order conditions are fulfilled then s' is a Nash Euqilibrium. Now one can repeat this procedure with s' instead of s^*. Therefore, it is obvious that there exist an infinite amount of equilibria.

Q.E.D.

Proof of Lemma 3:

We will first prove that the described equilibrium is indeed a Nash Equilibrium and afterwards show that this is the only Nash-Equilibrium by ruling out all the others equilibria.

1. Is $\{(x_1^*, y_1^*|_{y_2=0}), (x_2^*, y_2^* = 0)\}$ with $y_1^*|_{y_2=0} > y_2^*|_{y_1=0}$ a Nash Equilibrium?

The FOCs for the firms with respect to the public good are:

$$\frac{\partial \pi_1}{\partial y_1} = x_1 * \frac{\partial p}{\partial y_1} - \frac{\partial K_1^y}{\partial y_1} = 0$$

$$\frac{\partial \pi_2}{\partial y_2} = x_2 * \frac{\partial p}{\partial y_2} - \frac{\partial K_2^y}{\partial y_2} = 0$$

Denote by $y_1^*|_{y_2}$ ($y_2^*|_{y_1}$) the solution of $\frac{\partial \pi_1}{\partial y_1}$ ($\frac{\partial \pi_1}{\partial y_1}$) given a certain value of y_2 (y_1). Assume (without loss of generality) that $y_1^*|_{y_2=0} > y_2^*|_{y_1=0}$. Because $\frac{\partial p}{\partial y_i}$ decreases in y_i if follows:

$$x_2 * \frac{\partial p}{\partial y_2^*|_{y_1=0}} - \frac{\partial K_2^y}{\partial y_2^*|_{y_1=0}} = 0 > x_2 * \frac{\partial p}{\partial y_1^*|_{y_2=0}} - \frac{\partial K_2^y}{\partial y_1^*|_{y_2=0}}$$

This is the case because $\frac{\partial K_2^y}{\partial y_2^*|_{y_1=0}} \leq \frac{\partial K_2^y}{\partial y_1^*|_{y_2=0}}$ is true if the marginal costs are constant or dependent (whereby in the dependent case the marginal costs are by Equation 1 and Assumption $A5$ constant or increasing in Y).

Hence $\{(x_1^*, y_1^*|_{y_2=0}), (x_2^*, y_2^* = 0)\}$ is a Nash-Equilibrium.

2. Is $\{(x_1^*, y_1^*|_{y_2=0}), (x_2^*, y_2^* = 0)\}$ with $y_1^*|_{y_2=0} > y_2^*|_{y_1=0}$ the unique Nash Equilibrium?

First, let us look how y_1^* reacts given one knows that y_2 changes:

By total differentiation of $\frac{\partial \pi_1}{\partial y_1} = x_1 * \frac{\partial p}{\partial y_1} - \frac{\partial K_1^y(y_1)}{\partial y_1} = 0$ one gets:

$$[x_1 * \frac{\partial^2 p}{\partial y^2} - \frac{\partial^2 K_1^y}{\partial y_1^2}]dy_1 + [x_1 * \frac{\partial^2 p}{\partial y^2} - \frac{\partial^2 K_1^y}{\partial y_1 \partial y_2}]dy_2 = 0$$

$$dy_1 = -dy_2.$$

One can easily do the same exercise with y_2^* and y_1 which delivers the equivalent result.

Hence, we have a 1:1 reduction.

Now we are going to rule out all other possible equilibria.

In a Nash Equilibrium it must be true

$$\frac{\partial \pi_1}{\partial y_1} = x_1 * \frac{\partial p}{\partial y_1} - \frac{\partial K_1^y(y_1)}{\partial y_1} \leq 0.$$

Otherwise firm 1 has an incentive to produce more y_1. Because $y_2^*|_{y_1=0}$ solves

$$x_2 * \frac{\partial p}{\partial y_2^*|_{y_1=0}} - \frac{\partial K_2^y(y_1)}{\partial y_2^*|_{y_1=0}} = 0$$

firm 2 will never produce more than $y_2^*|_{y_1=0}$, so that

$$y_2^{max} = y_2^*|_{y_1=0}.$$

This leads to the fact that firm 1 at least produces

$$y_1^{min} = y_1^*|_{y_2=0} - y_2^{max}.$$

But this leads once again to the fact that firm 2 produces maximal

$$y_2^{max'} = y_2^{max} - y_1^{min}.$$

Knowing this, firm 1 will at least produce

$$y_1^{min'} = y_1^*|_{y2=0} - y_2^{max'} > y_1^{min}.$$

Continuing, one sees that y_1^{min} converges to $y_1^{max} = y_1^*|_{y2=0}$ and y_1^{max} converges to 0.

Hence, $\{(x_1^*, y_1^*|_{y2=0}), (x_2^*, y_2^* = 0)\}$ with $y_1^*|_{y2=0} > y_2^*|_{y1=0}$ is the unique Nash Equilibrium.

Q.E.D.

Proof of Lemma 4:

Profit function of firm i:

$$\pi_i = x_i * p(x_i, y_i, X_{-i}, Y_{-i}) - K(x_i) - K(y_i)$$

$$\pi_i = \pi_i(x_i, y_i, X_{-i}, Y_{-i})$$

FOCs:

$$\frac{\partial \pi_i(x_i, y_i, X_{-i}, Y_{-i})}{\partial x_i} = 0 \quad (46)$$

$$\frac{\partial \pi_i(x_i, y_i, X_{-i}, Y_{-i})}{\partial y_i} = 0 \quad (47)$$

(46) and (47) are a system of two equations with two endogenous and two exogenous variables. The solution will be given by the conditions:

$$f^1(\overbrace{x^*, y^*}^{endogenous}, \overbrace{X_{-i}, Y_{-i}}^{exogenous}) = 0 \quad (48)$$

$$f^2(\overbrace{x^*, y^*}^{endogenous}, \overbrace{X_{-i}, Y_{-i}}^{exogenous}) = 0 \quad (49)$$

It is possible to solve Equations (48) and (49) as differentiable functions of X_{-i} and Y_{-i}, so $x^*(X_{-i}, Y_{-i})$ and $y^*(X_{-i}, Y_{-i})$.

Inserting into (48) and (49) gives:

$$f^1(x^*(X_{-i}, Y_{-i}), y^*(X_{-i}, Y_{-i}), X_{-i}, Y_{-i}) = 0 \tag{50}$$

$$f^2(x^*(X_{-i}, Y_{-i}), y^*(X_{-i}, Y_{-i}), X_{-i}, Y_{-i}) = 0 \tag{51}$$

Regarding (50) and (51) as identities, we differentiate through with respect to X_{-i}, to obtain

$$\frac{\partial f^1}{\partial x^*}\frac{\partial x^*}{\partial X_{-i}} + \frac{\partial f^1}{\partial y^*}\frac{\partial y^*}{\partial X_{-i}} + \frac{\partial f^1}{\partial X_{-i}} = 0 \tag{52}$$

$$\frac{\partial f^2}{\partial x^*}\frac{\partial x^*}{\partial X_{-i}} + \frac{\partial f^2}{\partial y^*}\frac{\partial y^*}{\partial X_{-i}} + \frac{\partial f^2}{\partial X_{-i}} = 0 \tag{53}$$

The partial derivatives are all evaluated at the given equilibrium point and so can be regarded as given numbers. Rewriting these equations as the linear system

$$\begin{bmatrix} \partial f^1/\partial x_i^* & \partial f^1/\partial y_i^* \\ \partial f^2/\partial x_i^* & \partial f^2/\partial y_i^* \end{bmatrix} \begin{bmatrix} \partial x^*/\partial X_{-i} \\ \partial y^*/\partial X_{-i} \end{bmatrix} = \begin{bmatrix} -\partial f^1/\partial X_{-i} \\ -\partial f^2/\partial X_{-i} \end{bmatrix}$$

The system is solvable if the determinant

$$|D| = \partial f^1/\partial x_i^* * \partial f^2/\partial y_i^* - \partial f^1/\partial y_i^* * \partial f^2/\partial x_i^* \neq 0 \tag{54}$$

This is always the case because of Assumption A7 which ensure that, $|D|$ is always positive and therefore 54 is always true.

Applying Cramer's rule we have the solutions

$$\frac{\partial x_i^*}{\partial X_{-i}} = \frac{\begin{vmatrix} -\partial f^1/\partial X_{-i} & \partial f^1/\partial y_i^* \\ -\partial f^2/\partial X_{-i} & \partial f^2/\partial y_i^* \end{vmatrix}}{|D|} = \frac{-\partial f^1/\partial X_{-i} * \partial f^2/\partial y_i^* + \partial f^2/\partial X_{-i} * \partial f^1/\partial y_i^*}{|D|} \tag{55}$$

$$\frac{\partial y_i^*}{\partial X_{-i}} = \frac{\begin{vmatrix} \partial f^1/\partial x_i^* & -\partial f^1/\partial X_{-i} \\ \partial f^2/\partial x_i^* & -\partial f^2/\partial X_{-i} \end{vmatrix}}{|D|} = \frac{-\partial f^1/\partial x_i^* * \partial f^2/\partial X_{-i} + \partial f^2/\partial x_i^* * \partial f^1/\partial X_{-i}}{|D|} \tag{56}$$

Through the same way we can derive the comparative-statics effects of changes in Y_{-i}:

$$\frac{\partial x_i^*}{\partial Y_{-i}} = \frac{\begin{vmatrix} -\partial f^1/\partial Y_{-i} & \partial f^1/\partial y_i^* \\ -\partial f^2/\partial Y_{-i} & \partial f^2/\partial y_i^* \end{vmatrix}}{|D|} = \frac{-\partial f^1/\partial Y_{-i} * \partial f^2/\partial y_i^* + \partial f^2/\partial Y_{-i} * \partial f^1/\partial y_i^*}{|D|} \tag{57}$$

$$\frac{\partial y_i^*}{\partial Y_{-i}} = \frac{\begin{vmatrix} \partial f^1/\partial x_i^* & -\partial f^1/\partial Y_{-i} \\ \partial f^2/\partial x_i^* & -\partial f^2/\partial Y_{-i} \end{vmatrix}}{|D|} = \frac{-\partial f^1/\partial x_i^* * \partial f^2/\partial Y_{-i} + \partial f^2/\partial x_i^* * \partial f^1/\partial Y_{-i}}{|D|} \tag{58}$$

Q.E.D.

Proof of Proposition 1:

The profit function of firm i can be written as:

$$\pi_i = \pi_i(x_i^*(X_{-i}, Y_{-i}), y_i^*(X_{-i}, Y_{-i}), X_{-i}, Y_{-i}). \tag{59}$$

Total differentiation leads to

$$d\pi_i = [\overbrace{\frac{\partial \pi_i}{\partial x_i}\frac{\partial x_i^*}{\partial X_{-i}}}^{=0} + \overbrace{\frac{\partial \pi_i}{\partial y_i}\frac{\partial y_i^*}{\partial X_{-i}}}^{=0} + \frac{\partial \pi_i}{\partial X_{-i}}]dX_{-i} + [\overbrace{\frac{\partial \pi_i}{\partial x_i}\frac{\partial x_i^*}{\partial Y_{-i}}}^{=0} + \overbrace{\frac{\partial \pi_i}{\partial y_i}\frac{\partial y_i^*}{\partial Y_{-i}}}^{=0} + \frac{\partial \pi_i}{\partial Y_{-i}}]dY_{-i}. \tag{60}$$

Therefore one can distinguish between direct and indirect effects on the profit. The direct effect directly influence the profit whereby the indirect effect works over changing the production decision of firm i. But if the firm is in a maximum these indirect effects are always zero because the first order derivative should be equal to zero.

Hence, there are only direct effects:

$$d\pi_i = \frac{\partial \pi_i}{\partial X_{-i}} dX_{-i} + \frac{\partial \pi_i}{\partial Y_{-i}} dY_{-i}. \tag{61}$$

Calculation these direct effects lead to

$$\pi_i = x_i * p(x_i, y_i, X_{-i}, Y_{-i}) - K(x_i) - K(y_i) \tag{62}$$

$$d\pi_i = x_i * \frac{\partial p}{\partial X_{-i}} dX_{-i} + x_i * \frac{\partial p}{\partial Y_{-i}} dY_{-i}. \tag{63}$$

Therefore, it gets obvious that the change of the price determines the effect on the profit.

Q.E.D.

<u>The Effects of an exogenous change in Y_{-i}:</u>

Case 1:
1. If $\frac{\partial^2 R}{\partial x \partial y} = [\frac{\partial p}{\partial y_i} + x_i \frac{\partial^2 p}{\partial x_i \partial y_i}] = 0 \rightarrow \frac{\partial x_i^*}{\partial Y_{-i}} = 0$

- The impact of x on the revenue is independent of y
- If I was in the optimum before, I remain in the optimum. Therefore I should not adjust my production.

1.1 If $\frac{\partial^2 R}{\partial y^2} = x * \frac{\partial^2 p}{\partial y^2} = 0$ and $\frac{\partial^2 K(y)}{\partial y^2} > 0 \rightarrow \frac{\partial y_i^*}{\partial Y_{-i}} = 0$
Reason:

- We know from above that we are not going to change our production of the private good
- The impact of one unit of the public good on the price is constant
- Although there is an additional production of the public good we are in our optimum and should not adjust our production.

1.2 If $\frac{\partial^2 R}{\partial y^2} = x * \frac{\partial^2 p}{\partial y^2} < 0$ and $\frac{\partial^2 K(y)}{\partial y^2} > 0$ then we know due the cross-derivative condition that it must be $\frac{\partial y_i^*}{\partial Y_{-i}} < 0$

- The impact of one unit public good on the price is now lower
- The marginal costs are increasing
- The production of the private good remains constant (see above)
- Therefore the firm has an incentive to reduce its production of the public good

1.3 If $\frac{\partial^2 R}{\partial y^2} = x * \frac{\partial^2 p}{\partial y^2} < 0$ and $\frac{\partial^2 K(y)}{\partial y^2} = 0$ then we know due to the third second order condition that it must be that $\frac{\partial y_i^*}{\partial Y_{-i}} < 0$

- The additional production of the public good, holding everything else constant, reduces the impact of one unit of the public good on the price.
- The marginal costs of producing the public good are constant.
- This leads to a 1:1 reduction of the production of the public good
- Therefore we have perfect substitution.

Case 2:
2. If $\frac{\partial^2 R}{\partial y^2} = x * \frac{\partial^2 p}{\partial y^2} < 0$, $\frac{\partial^2 K(y)}{\partial y^2} > 0$ and $\frac{\partial^2 R}{\partial x \partial y} = [\frac{\partial p}{\partial y_i} + x_i \frac{\partial^2 p}{\partial x_i \partial y_i}] > 0 \rightarrow \frac{\partial x_i^*}{\partial Y_{-i}} > 0$

- Suppose I reduce my production of the public good by 1:1 with the additional production of the public good. → The impact of one additional unit of the public good is as big as before.

- Given that my marginal costs are now lower I have an incentive to increase my production of the public good a little bit.

- Therefore I am not going to substitute the public good production by 1:1 and so the total amount of the public good increases → This gives me an incentive to increase my production of the private good.

$\rightarrow \frac{\partial y_i^*}{\partial Y_{-i}} \lesseqgtr 0$

- We know that under these circumstances we are going to increase our production of the private good. This gives me an incentive to increase my production of the public good.

- Due to the fact that the impact of the public good on the price is decreasing and the marginal costs of the public are increasing I have an incentive to reduce my production of the public good. → It depends on which effect is bigger.

Case 3:

3. If $\frac{\partial^2 R}{\partial y^2} = x * \frac{\partial^2 p}{\partial y^2} = 0$, $\frac{\partial^2 K(y)}{\partial y^2} > 0$ and $\frac{\partial^2 R}{\partial x \partial y} = [\frac{\partial p}{\partial y_i} + x_i \frac{\partial^2 p}{\partial x_i \partial y_i}] > 0 \rightarrow \frac{\partial x_i^*}{\partial Y_{-i}} > 0$

- The impact of one additional unit of the public good on the price is always constant.

- Therefore the additional production of the public good increases the price of the private good and gives me an incentive to increase my production of the private good.

- $\frac{\partial R^2}{\partial x \partial y} > 0$ → My old production level of the public good was optimal for the old level of my private good production → It cannot be good to decrease my production of the public good

$\rightarrow \frac{\partial y_i^*}{\partial Y_{-i}} > 0$

- The impact of one unit of the public good on the price does not change.
- The marginal costs of the public good production are increasing.
- There is no direct effect on the incentives to produce the public good.
- But due to the analysis above we know that the production of the private good increases under these circumstances.
- Hence, the firm is going to increase his production of the public good.

Case 4:
If $\frac{\partial^2 R}{\partial y^2} = x * \frac{\partial^2 p}{\partial y^2} < 0$, $\frac{\partial^2 K(y)}{\partial y^2} = 0$ and $\frac{\partial^2 R}{\partial x \partial y} = [\frac{\partial p}{\partial y_i} + x_i \frac{\partial^2 p}{\partial x_i \partial y_i}] > 0$ then $\frac{\partial x_i^*}{\partial Y_{-i}} = 0$

- Given my production of the private good there is an optimal amount of the public good where the marginal benefits equal the marginal cost.
- Now in this case the marginal costs are constant.
- Therefore, through the additional production of the public good, I just decrease my production of the public good without changing my production of the private good.

$\rightarrow \frac{\partial y_i^*}{\partial Y_{-i}} < 0$ (technial: Due to the SOC with cross-derivative)

- The additional production of the public good, holding everything else constant, reduces the impact of one unit of the public good on the price.
- The marginal costs of producing the public good are constant.
- This leads to a 1:1 reduction of the production of the public good

- Looking at the production of the private good we see that it does not change, because the positive effect of the additional amount of the public good is as high as the negative effect due to the reduction of the production of the public good.

- Therefore we have perfect substitution.

<u>**Proof of Proposition 6:**</u>

For the first part of the proof we take the derivative of X with respect to N and look at the sign.

$$\frac{\partial X}{\partial N} = \frac{\partial[N * x(N)]}{\partial N} = N * \frac{\partial x}{\partial N} + x \tag{64}$$

$$\frac{\partial X}{\partial N} = -N * \frac{A}{[1 + N - \frac{1}{2f}N + 2d]^2} * (1 - \frac{1}{2f}) + \frac{A}{1 + N - \frac{1}{2f}N + 2d} \tag{65}$$

$$\frac{\partial X}{\partial N} = \frac{A}{1 + N - \frac{1}{2f}N + 2d}(-N * \frac{1}{1 + N - \frac{1}{2f}N + 2d} * (1 - \frac{1}{2f}) + 1) \tag{66}$$

$$\frac{\partial X}{\partial N} = \frac{A * [1 + 2d]}{(1 + N - \frac{1}{2f}N + 2d)^2} > 0 \tag{67}$$

By Proposition 5 we know that the private and the public good are always individually produced in the same ratio. Summing up, we see that the total amounts produced must also have the same ratio. If X is always increasing, then Y is also always increasing.

Q.E.D.

<u>**Proof of Proposition 7:**</u>

For the proof we use the fact that $y = \frac{1}{2f}x$ (26) and rewrite the demand function (20) as follows:

$$p = A - X + Y = A - X(N) + \frac{1}{2f}X(N) = A + X(N)(-1 + \frac{1}{2f}) \tag{68}$$

$$p = A + N * \frac{A}{a + N - \frac{1}{2f}N + 2d}(-1 + \frac{1}{2f}) \tag{69}$$

Now we can take the first derivative of p with respect to N and look at the sign.

$$\frac{\partial p}{\partial N} = (\,-1 + \frac{1}{2f}) * \frac{A(1+2d)}{[1 + N - \frac{1}{2f}N + 2d]^2} \tag{70}$$

Therefore

- $\frac{\partial p}{\partial N} < 0$ if $f > 0.5$
- $\frac{\partial p}{\partial N} = 0$ if $f = 0.5 \rightarrow$ p=A
- $\frac{\partial p}{\partial N} > 0$ if $f < 0.5$

Q.E.D.

Proof of Proposition 8:

$$\pi_i = p * x_i - dx^2 - f[\frac{1}{2f}x_i]^2 = p * x_i - x_i^2(d + \frac{1}{4f}) \tag{71}$$

$$\pi_i = (A + N * \frac{A(-1 + \frac{1}{2f})}{1 + N - \frac{1}{2f}N + 2d}) * \frac{A}{1 + N - \frac{1}{2f}N + 2d} - (\frac{A}{1 + N - \frac{1}{2f}N + 2d})^2(d + \frac{1}{4f}) \tag{72}$$

$$\pi_i = \frac{A^2(1 + d - \frac{1}{4f})}{(1 + N - \frac{1}{2f}N + 2d)^2} \tag{73}$$

$$\frac{\partial \pi_i}{\partial N} = 2A^2 * \frac{1}{(1+N-\frac{1}{2f}N+2d)^3}(1-\frac{1}{2f})(-1-d+\frac{1}{4f}) \tag{74}$$

The last term $-1-d+\frac{1}{4f}$ is always negative because of the SOC in the maximization problem. We know by Equation (??) that it must be true that

$$1^2 < (-2-2d)(-2f) \tag{75}$$

Solving for f yields to

$$f > \frac{1}{4+4d} \tag{76}$$

Assuming $-1-d+\frac{1}{4f} < 0$ and solving for f leads to:

$$f > \frac{1}{4+4d} \tag{77}$$

Therefore we have to analyze the first and the second term.

Case 1: $f = 0.5$

Then $1-\frac{1}{2f} = 0$ and therefore $\frac{\partial \pi}{\partial N} = 0$.

Case 2: $f > 0.5$

Then $1-\frac{1}{2f} > 0$ and the sign of $\frac{\partial \pi}{\partial N}$ depends on N because of the term $(1+N-\frac{1}{2f}N+2d)$.

This term is always positive:

$$1+N-\frac{1}{2f}N+2d = 1+N(1-\frac{1}{2f})+2d \tag{78}$$

Therefore $\frac{\partial \pi}{\partial N} < 0$ for all $N > 0$

Case 3: $f < 0.5$

Then $1 - \frac{1}{2f} < 0$ and the sign of $\frac{\partial \pi}{\partial N}$ depends on N because of the term $(1 + N - \frac{1}{2f}N + 2d)$.

This term is zero if

$$N^* = \frac{2d + 1}{\frac{1}{2f} - 1} > 0 \tag{79}$$

The slope of $(1 + N - \frac{1}{2f}N + 2d)$ with respect to N is

$$\frac{\partial(1 + N - \frac{1}{2f}N + 2d)}{\partial N} = 1 - \frac{1}{2f} < 0 \tag{80}$$

Therefore $\frac{\partial \pi}{\partial N} > 0$ in the relevant area where $N < \overline{N}$.

Q.E.D.

Proof of Proposition 9:

First we calculate the effect of more competition on the consumer surplus.

$$CS = (A + Y - p) * X * 0.5 = 0.5 * X^2 \tag{81}$$

We see that the consumer surplus always increases. In the case of $f \geq 0.5$, this is quite obvious. The total sold amount of the private good increases and the price does not raise. If the weight of the public good cost function is low, than the price for the private good increases and the total production of both goods also increases. Nevertheless the consumers benefit from more competition, because due to the higher production of the public good, their willingness to pay increases and therefore their surplus increases even though the price raises.

The total surplus is the sum of the companies profits and the consumer surplus.

$$TS = N * \pi_i + CS \tag{82}$$

To see the reaction of the total surplus caused by a variation in N, we take the first derivative of TS with respect to N and look at the sign.

$$\frac{\partial TS}{\partial N} = \frac{\partial(N * \pi_i)}{\partial N} + \frac{\partial CS}{\partial N} \tag{83}$$

$$\pi_i * N = N * \frac{A^2(1 + d - \frac{1}{4f})}{(1 + N - \frac{1}{2f} * N + 2d)^2} \tag{84}$$

$$CS = \frac{1}{2} * \frac{(AN)^2}{(1 + N - \frac{1}{2f} * N + 2d)^2} \tag{85}$$

$$\frac{\partial TS}{\partial N} = \frac{A^2(1 + d - \frac{1}{4f})(1 - N + \frac{1}{2f}N + 2d)}{(1 + N - \frac{1}{2f}N + 2d)^3} + \frac{A^2 N(1 + 2d)}{(1 + N - \frac{1}{2f}N + 2d)^3} \tag{86}$$

$$\frac{\partial TS}{\partial N} = \frac{A^2}{(1 + N - \frac{1}{2f}N + 2d)^3}[(1 + d - \frac{1}{4f})(1 - N + \frac{1}{2f}N + 2d) + N(1 + 2d)] \tag{87}$$

$$\frac{\partial TS}{\partial N} = \frac{A^2(1 - \frac{1}{4f} + 3d - \frac{1}{2}\frac{d}{f} + N\frac{3}{4f} - \frac{1}{8f^2}N + 2d^2 + \frac{1}{2f}dN + dN)}{\left(1 + N - \frac{1}{2f}N + 2d\right)^3} \tag{88}$$

If $f \geq 0.5$ then the denominator and the nominator is always positive. If $f < 0.5$ then we know that π_i and CS are increasing and therefore the total social surplus must also increase.

Q.E.D.

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